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Non-local potentials with LS terms in algebraic scattering theory

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Abstract. The group theoretical analysis of Coulomb scattering based on the $SO(3,1)$ group is revisited. Using matrix-valued differential operators, modifying the angular momentum and the Runge–Lenz vector used hitherto for the realization of the $so(3,1)$ (Lorentz) algebra, we obtain a three-dimensional solvable two-channel scattering problem. The interaction term besides the Coulomb potential contains a non-local potential of LS-type. Using the momentum representation the S-matrix can be calculated analytically. By employing a canonical transformation, another solvable three-dimensional scattering problem is found, in agreement with the expectations of algebraic scattering theory. The potential in this case is of Pöschl–Teller type with an LS term. It is also pointed out that our matrix-valued realization of the $so(3,1)$ algebra can be cast to an instructive form with the help of $su(2)$ gauge fields. An interesting connection between gauge transformations and supersymmetry transformations of supersymmetric quantum mechanics is also observed. These results enable us to construct other solvable scattering problems by using $su(2)$ gauge transformations.

1. Introduction

Since the seminal work of Pauli [1], Fock [2] and Bargmann [3] providing a group theoretical description of the Coulomb problem based on the dynamical symmetry group $SO(4)$, group theoretical methods has widely been applied to bound-state problems. However, until the advent of algebraic scattering theory (AST) [4], systematic group theoretical investigations of scattering problems had still been missing. It was only after AST has made its debut in physics that many interesting applications of group theory to the solution of scattering problems appeared.

In AST the quadratic Casimir of a *non-compact* group G is related to the Hamiltonian H of some scattering problem. Since, unlike in the *compact* case, G has unitary irreducible representations characterized by a continuous set of values, it is possible to relate the continuous set of eigenvalues of H (the scattering energy) to such values. By identifying the symmetry group of the interaction free (asymptotic) region, and using the theory of group contractions and expansions AST was capable of determining the most general form of the S-matrix for the fixed dynamical *non-compact* symmetry group G.

In AST the calculation of the S-matrix is performed with no recourse to any special coordinate realization of the generators of the Lie algebra $\boldsymbol{g}$ of G. Hence, no explicit form of the scattering Hamiltonian is usually written down. However, some authors [5–7] stressed the physical relevance of finding interesting coordinate realizations of $\boldsymbol{g}$, in order to extract the interaction terms (potentials) governing the scattering process. In this approach the general form of the algebraic S-matrix for the dynamical symmetry group G can further be specialized by an explicit calculation of the unknown functions not fixed by AST.

For the Coulomb problem—our main concern here—the non-compact group in question is $SO(3,1)$ (the proper orthochronous Lorentz group) [8]. The group $SO(3,1)$ ($SO(4)$) describes the continuous (discrete) $E > 0$ ($E < 0$) part of the spectrum. The continuously changing value f of direct group theoretical meaning turns out to be $f = \eta = Z_1 Z_2 e^2/k$, where $2E = k^2$ ($\hbar = m = 1$) and η is the Sommerfeld parameter. The scattering Hamiltonian H is some appropriately chosen function of the quadratic Casimir of $SO(3,1)$. Choosing the six generators of $so(3,1)$ (the Lie algebra of $SO(3,1)$) to be the three components of the angular momentum $\boldsymbol{L}$ and the three components of a vector operator $\boldsymbol{K}$ related to the components of the Runge–Lenz vector, the quadratic Casimir is $\mathcal{C} = \boldsymbol{L}^2 - \boldsymbol{K}^2$. Then its relation to the Coulomb Hamiltonian is [9]

$$H = -\frac{\eta^2}{\mathcal{C}+1}E. \tag{1.1}$$

Identifying the asymptotic algebra as the Lie algebra of the Euclidean group $E(3)$, the purely algebraic method yields the S-matrix [8, 9]

$$S_l = \frac{\Gamma(l+1+\mathrm{i}\eta)}{\Gamma(l+1-\mathrm{i}\eta)} \tag{1.2}$$

where l is the value of the angular momentum.

In this paper by employing a new coordinate realization we will modify the generators $\boldsymbol{L}$ and $\boldsymbol{K}$ commuting with the Hamiltonian of the Coulomb problem. The idea is to add matrix-valued terms to them. We demand that the resulting new generators $\boldsymbol{J}$ and $\boldsymbol{M}$ should satisfy the commutation relations of the $so(3,1)$ algebra. In order to achieve this we employ a canonical transformation yielding *first-order* differential operators for both $\boldsymbol{L}$ and $\boldsymbol{K}$. (Notice that $\boldsymbol{K}$ is originally a *second-order* differential operator since it is related to the Runge–Lenz vector.) In this canonically transformed realization we can quickly find the appropriate modifications. After the inverse canonical transformation we obtain a *non-local* realization for the $so(3,1)$ algebra. Of course, these new generators $\boldsymbol{J}$ and $\boldsymbol{M}$ are commuting with a Hamiltonian with a *modified Coulomb* potential. This realization will be introduced in section 2. With this new realization at hand we calculate the Casimir operator $\mathcal{C}$ in section 3. First, we choose the representation content of the matrix-valued modification to be that of a particle with spin-$\frac{1}{2}$. The trick is again to use the canonically transformed realization which is much easier to handle. We also emphasize here the role played by the other Casimir operator $\mathcal{C}' = \boldsymbol{J}\boldsymbol{M}$ which is non-zero. (The role of Casimir operators in AST other than the quadratic Casimir has been clarified in [10] and [11].) In the canonically transformed realization the interaction term can easily be identified and the resulting S-matrix can be calculated. The potential in this case is of Pöschl–Teller type with an LS term.

In section 4 we transform back the Casimir operators to the *non-local* realization. Using the Casimir operator $\mathcal{C}' = \boldsymbol{J}\boldsymbol{M}$ we can extract a non-local interaction term of LS-type. Since the interaction term also commutes with the operator of parity, the S-matrix of the resulting two-channel scattering problem is diagonal in the basis diagonalizing the LS term. In section 5, by using the momentum representation, we calculate the S-matrix for the resulting scattering problem. The geometric meaning of our matrix-valued realization will be clarified in section 6. Here we point out that this realization for the $so(3,1)$ algebra can be cast in an instructive form with the help of $su(2)$ gauge fields. Armed with this observation, we can derive other solvable three-dimensional scattering problems by gauge transforming the original scattering problems with conveniently chosen $su(2)$ gauge transformations. The conclusions and some comments are left for section 7.

2. A matrix-valued realization for $so(3,1)$

Let us start with the usual realization of the $so(3,1)$ (Lorentz) algebra in terms of the angular momentum $\boldsymbol{L}$ and the Runge–Lenz vector $\boldsymbol{K}'$,

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} \qquad \boldsymbol{K}' = \frac{1}{2}(\boldsymbol{P} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{P}) + \frac{Z_1 Z_2 e^2}{R}\boldsymbol{R}. \tag{2.1}$$

These vector operators commute with the Coulomb Hamiltonian

$$H = \frac{1}{2}P^2 + \frac{Z_1 Z_2 e^2}{R} \tag{2.2}$$

(in the following we set $\hbar = m = 1$). For scattering states the total energy $\boldsymbol{E}$ is positive. Since H commutes with our generators we can assume that they act merely on energy eigensubspaces. In this case we can renormalize our operators $\boldsymbol{K}'$ by setting

$$\boldsymbol{K} = \sqrt{\frac{1}{2H}}\boldsymbol{K}'. \tag{2.3}$$

The commutation relations of $\boldsymbol{L}$ and $\boldsymbol{K}$ are now in the form of an $so(3,1)$ algebra:

$$[L_i, L_j] = \mathrm{i}\varepsilon_{ijk}L_k \qquad [L_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k \qquad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}L_k \tag{2.4}$$

where $i, j, k = 1, 2, 3$.

Note that by virtue of the relation $\frac{1}{2}(\boldsymbol{P} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{P}) = \boldsymbol{R}P^2 - \boldsymbol{P}(\boldsymbol{R}\boldsymbol{P})$ on an eigensubspace of energy E the operator K takes the following form [6]:

$$\boldsymbol{K} = \frac{1}{\sqrt{2E}}\left(\frac{1}{2}\boldsymbol{R}P^2 - \boldsymbol{P}(\boldsymbol{R}\boldsymbol{P})\right) + \sqrt{\frac{2E}{2}}\boldsymbol{R}. \tag{2.5}$$

We see from (2.5) that $\boldsymbol{K}$ (unlike $\boldsymbol{L}$) is a second-order differential operator. Moreover, the presence of the factors $\sqrt{2E}$ is also disturbing. Hence, in order to easily find some matrix-valued modification of $\boldsymbol{L}$ and $\boldsymbol{K}$ we employ a canonical transformation of the following form [6]:

$$\boldsymbol{R} \mapsto \frac{1}{\sqrt{2E}}\boldsymbol{P} \qquad \boldsymbol{P} \mapsto -\sqrt{2E}\boldsymbol{R}. \tag{2.6}$$

Hence the canonically transformed realization for $so(3,1)$ is spanned by the operators

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} \qquad \boldsymbol{K} = \tfrac{1}{2}\boldsymbol{P}(1 + R^2) - \boldsymbol{R}(\boldsymbol{P}\boldsymbol{R}). \tag{2.7}$$

This is a realization in terms of *first-order* differential operators.

Now we try to find matrix-valued first-order differential operators $\boldsymbol{J}$ and $\boldsymbol{M}$ by adding matrix-valued terms, not containing the differential operator $\boldsymbol{P}$, to $\boldsymbol{L}$ and $\boldsymbol{K}$. Among the many possible realizations we restrict our attention to those for which $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, where $\boldsymbol{S}$ are $(2s+1) \times (2s+1)$ spin matrices in the representation of the Lie-algebra $so(3)$ labelled by s. This choice is also dictated by simplicity, and our desire to arrive at a group theoretical description of potentials with LS terms. We would like to satisfy the commutation relations

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i, M_j] = \mathrm{i}\varepsilon_{ijk}M_k \qquad [M_i, M_j] = -\mathrm{i}\varepsilon_{ijk}J_k. \tag{2.8}$$

The first relation of (2.8) is trivially satisfied. Moreover, one can easily convince oneself that the combination $\boldsymbol{M} = \boldsymbol{K} + F(R)\boldsymbol{S} \times \boldsymbol{R}$ satisfies the second. Hence we are left with the determination of the function $F(R)$. This function can be determined from the third set of commutation relations. A straightforward calculation shows that

$$[M_i, M_j] = \mathrm{i}\varepsilon_{ijk}\left(\left(\left(\frac{1}{2}R(R^2-1)F' - F\right)S_k - L_k\right) + \left(F^2 - F - \frac{F'}{2R}(R^2-1)\right)(\boldsymbol{S}\boldsymbol{R})R_k\right). \tag{2.9}$$

We can see from this equation that the third relation of (2.8) is satisfied provided F satisfies the differential equations

$$F - \frac{1}{2}R(R^2-1)F' = 1 \qquad \text{and} \qquad F^2 - F - \frac{F'}{2R}(R^2-1) = 0. \tag{2.10}$$

It is easy to show that the solutions to these equations are $F(R) = 1$ and $F(R) = 1/R^2$; hence our matrix-valued realization is

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \qquad \boldsymbol{M} = \boldsymbol{K} + F(R)\boldsymbol{S}\times\boldsymbol{R} \qquad F(R) = \begin{cases} 1 \\ 1/R^2. \end{cases} \tag{2.11}$$

After transforming back with the help of the inverse of the canonical transformation (2.6) we obtain

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \qquad \boldsymbol{M} = \boldsymbol{K} + F_E(P)\boldsymbol{P}\times\boldsymbol{S} \qquad F_E(P) = \begin{cases} 1/\sqrt{2E} \\ \sqrt{2E}/P^2. \end{cases} \tag{2.12}$$

It is important to stress that in this case the eigenvalue E corresponds to a (yet unknown) Hamiltonian commuting with $\boldsymbol{J}$ and $\boldsymbol{M}$. Moreover, realization (2.12) for $F_E(P) = \sqrt{2E}/P^2$ is non-local due to the presence of the operator P^{-2}. Since $P^{-2} = -\triangle$, with the help of the Green function of the Laplace operator its action on a function can be written as

$$(P^{-2}\psi)(\boldsymbol{R}) = \frac{1}{4\pi}\int \mathrm{d}^3R' \frac{\psi(\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|}. \tag{2.13}$$

Since we are primarily interested in the possibility of obtaining non-local potentials, the discussion of the other solution $F_E(P) = 1/\sqrt{2E}$ will be deferred to section 6.

In closing this section we remark that constructing such matrix-valued realizations is motivated by the theory of induced representations. This theory tells us how to construct unitary (but generally reducible) representations of G starting from an unitary irreducible representation of some subgroup $\mathcal{H}$. In our case this subgroup of $SO(3,1)$ is the maximally compact subgroup $SO(3)$ corresponding to our spin degrees of freedom. In this context the interested reader is asked to consult [12] where the theory of induced representations has been used to construct the explicit form of modified symmetry generators for both compact and non-compact symmetry groups.

3. The scattering problem in the canonically transformed realization

In this section by calculating the Casimir operators $\mathcal{C} = \boldsymbol{J}^2 - \boldsymbol{M}^2$ and $\mathcal{C}' = \boldsymbol{J}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{J}$ using the canonically transformed realization (2.7) and (2.11) we derive the scattering potential compatible with the $SO(3,1)$ symmetry. Note in this respect that we have *two* Casimir operators corresponding to the fact that $SO(3,1)$ is a group of *rank two*. Had we used the usual realization in terms of $\boldsymbol{L}$ and $\boldsymbol{K}$ of (2.7) we would have obtained zero for the Casimir $\mathcal{C}'$. However, in this new realization $\mathcal{C}' \neq 0$; hence we have no reason for neglecting the possible relevance of this operator. Indeed, for the complete characterization of the scattering states, the eigenvalues of *both* Casimirs are needed [10].

Using the realization (2.7) and (2.11) straightforward calculation shows that

$$\mathcal{C} = \boldsymbol{J}^2 - \boldsymbol{M}^2 = -\frac{1}{4}[\boldsymbol{P}(1-R^2)]^2 + (\mathcal{M} + R^2)\frac{R^2-1}{2R^2} - \frac{3}{4} \tag{3.1a}$$

$$\mathcal{C}' = \boldsymbol{J}\boldsymbol{M} = \left(\frac{2\mathrm{i}}{R^2}\mathcal{M}\boldsymbol{S}\boldsymbol{R} + \boldsymbol{S}\boldsymbol{P}\right)\frac{1}{2}(1-R^2) \tag{3.1b}$$

where

$$\mathcal{M} \equiv 2\boldsymbol{L}\boldsymbol{S} + 1. \tag{3.2}$$

In this paper we only consider the simplest non-trivial case when the spin is $\frac{1}{2}$, i.e.

$$\boldsymbol{S} \equiv \tfrac{1}{2}\boldsymbol{\sigma}. \tag{3.3}$$

In this case $\mathcal{M} = \sigma\boldsymbol{L} + 1$, and it satisfies

$$\{\mathcal{M}, \boldsymbol{\sigma}\boldsymbol{R}\} = 0 \qquad \{\mathcal{M}, \boldsymbol{\sigma}\boldsymbol{P}\} = 0. \tag{3.4}$$

With the choice (3.3) using the identity $(\boldsymbol{\sigma}\boldsymbol{a})(\boldsymbol{\sigma}\boldsymbol{b}) = \boldsymbol{a}\boldsymbol{b} + \mathrm{i}(\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{\sigma}$ one can prove that

$$\mathcal{C}' = \frac{\mathrm{i}}{4}(\boldsymbol{\sigma}\boldsymbol{n})\left[\frac{\partial}{\partial\boldsymbol{R}} + \frac{\mathcal{M}+1}{R}\right](R^2-1) \tag{3.5}$$

where $\boldsymbol{n} \equiv \boldsymbol{R}/R$. After calculating $\mathcal{C}$ and the square of $\mathcal{C}'$ we find

$$\mathcal{C} + (2\mathcal{C}')^2 + \tfrac{3}{4} = 0. \tag{3.6}$$

Hence, $-\mathcal{C} - \frac{3}{4}$ is the square of $2\mathcal{C}'$. Now, following [10] we show that (3.6) fixes the representation content of the scattering states. The irreducible representations of $SO(3,1)$ capable of characterizing scattering states are classified by the pair (j_0, j_1), where $j_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and $j_1 = \mathrm{i}k$, $k \in \boldsymbol{R}_0^+$ [13]. According to AST, scattering states are labelled as $|j_0, j_1\rangle$. The action of the Casimir operators on this base is [13]

$$\mathcal{C}|j_0, j_1\rangle = (j_0^2 + j_1^2 - 1)|j_0, j_1\rangle \tag{3.7a}$$

$$\mathcal{C}'|j_0, j_1\rangle = -\mathrm{i}j_0 j_1|j_0, j_1\rangle. \tag{3.7b}$$

Using equations (3.7) together with the identity (3.6) we find the relation

$$(j_0^2 - \tfrac{1}{4})(j_1^2 - \tfrac{1}{4}) = 0 \tag{3.8}$$

so we can single out the states $(\pm\frac{1}{2}, \mathrm{i}k)$ transforming according to the (inequivalent) representations mirror-conjugated to each other. The next step is to extract the interaction term (potential) from the eigenvalue problem of the Casimir operators, i.e. to use equations (3.7a, b). Due to the relation (3.6) we can choose merely *one* of such operators. Let us choose the Casimir $\mathcal{C}'$ which is a first-order differential operator. Its square will result in a Schrödinger equation with some scattering potential. The form of $\mathcal{C}'$ after a coordinate transformation [6]

$$R(r) = \coth(r/2) \tag{3.9}$$

is

$$\frac{\mathrm{i}}{4}(\sigma\boldsymbol{n})\left[-2\sinh^2(r/2)\frac{\partial}{\partial r} + \frac{\mathcal{M}+1}{\coth(r/2)}\right]\frac{1}{\sinh^2(r/2)}. \tag{3.10}$$

Then we apply a similarity transformation

$$\mathcal{C}' \mapsto T^{-1}(r)\mathcal{C}'T(r) \qquad T(r) = r\sinh^2(r/2)\tanh(r/2) \tag{3.11}$$

to obtain

$$\frac{\mathrm{i}}{4}(\boldsymbol{\sigma n})\left[-\frac{\partial}{\partial \boldsymbol{r}}-\frac{1}{r}+\frac{\mathcal{M}}{\sinh r}\right]. \tag{3.12}$$

According to (3.6) four times the square of this operator gives (up to a constant) the quadratic Casimir related to the scattering Hamiltonian:

$$(2\mathcal{C}')^2=-\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}-\frac{\mathcal{M}}{\sinh^2 r}(\cosh r-\mathcal{M}). \tag{3.13}$$

Knowing that the scattering states labelled as $(\pm\frac{1}{2},\mathrm{i}k)$ are not discriminated by the operator $\mathcal{C}$ and using (3.7*b*) we can see that the Schrödinger equation is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\boldsymbol{L}^2}{\sinh^2 r}-\frac{\mathcal{M}}{2\cosh^2(r/2)}\right)\Psi_{\frac{1}{2},\mathrm{i}k,\lambda}=k^2\Psi_{\frac{1}{2},\mathrm{i}k,\lambda} \tag{3.14}$$

where, in order to specify this equation further, we have to solve the eigenvalue problem of the operator $\mathcal{M}$ acting merely on the angular part of the function $\Psi_{\frac{1}{2},\mathrm{i}k,l,\lambda}(r,\theta,\varphi)\equiv\mathcal{R}_{\frac{1}{2},\mathrm{i}k,l,\lambda}(r)\Phi_{l,\lambda,m}(\theta,\varphi)$. Since $H=-\mathcal{C}-\frac{3}{4}$ commutes with $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$, this suggests using the states (spinor harmonics)

$$\mathcal{Y}_{j-\frac{1}{2},j,m}=\frac{1}{\sqrt{2j}}\left(\sqrt{j+m}Y^{m-\frac{1}{2}}_{j-\frac{1}{2}}\chi_++\sqrt{j-m}Y^{m+\frac{1}{2}}_{j-\frac{1}{2}}\chi_-\right) \tag{3.15a}$$

$$\mathcal{Y}_{j+\frac{1}{2},j,m}=\frac{1}{\sqrt{2j+2}}\left(-\sqrt{j-m+1}Y^{m-\frac{1}{2}}_{j+\frac{1}{2}}\chi_++\sqrt{j+m+1}Y^{m+\frac{1}{2}}_{j+\frac{1}{2}}\chi_-\right) \tag{3.15b}$$

which are eigenstates of $\boldsymbol{L}^2$, $\boldsymbol{J}^2$ and J_3 expressed in terms of eigenstates of $\boldsymbol{L}^2$, L_3, and S_3. The action of $\mathcal{M}$ on the spinor harmonics is

$$\mathcal{M}\mathcal{Y}_{l,j,m}(\theta,\varphi)=\lambda\mathcal{Y}_{l,j,m}(\theta,\varphi) \tag{3.16}$$

with

$$\lambda=\pm(j+\tfrac{1}{2})=\begin{cases} l+1 & \text{for } j=l+\frac{1}{2}\\ -l & \text{for } j=l-\frac{1}{2}\end{cases} \tag{3.17}$$

and for $l=0$ the only possible value is $\lambda=1$. Hence, with the definition

$$\Phi_{l,\lambda,m}(\theta,\varphi)=\mathcal{Y}_{l,j,m}(\theta,\varphi) \tag{3.18}$$

we obtain the radial Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l(l+1)}{\sinh^2 r}-\frac{\lambda}{2\cosh^2(r/2)}\right)\mathcal{R}_{\frac{1}{2},\mathrm{i}k,\lambda}=k^2\mathcal{R}_{\frac{1}{2},\mathrm{i}k,\lambda}. \tag{3.19}$$

Hence the potential is

$$V=-\frac{l(l+1)}{2}\left(\frac{1}{r^2}-\frac{1}{\sinh^2 r}\right)-\frac{\lambda}{4\cosh^2(r/2)}. \tag{3.20}$$

Notice that the first term replaces the usual centrifugal term with $l(l+1)/2\sinh^2 r$, and the second term is a spin–orbit term. The radial equation (3.19) can easily be solved by noting that, by virtue of the relation $\boldsymbol{L}^2=\mathcal{M}(\mathcal{M}-1)$, it can be written in the form

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\lambda(\lambda-1)}{4\sinh^2(r/2)}-\frac{\lambda(\lambda+1)}{4\cosh^2(r/2)}\right)r\mathcal{R}_{\frac{1}{2},\mathrm{i}k,\lambda}(r)=k^2r\mathcal{R}_{\frac{1}{2},\mathrm{i}k,\lambda}(r) \tag{3.21}$$

which is of the form of a solvable problem known from [7, 10, 11]. The calculation of the S-matrix elements can be performed by noting that $[H, \boldsymbol{J}] = [H, J_3] = 0$, and parity is also conserved. Hence the S-matrix is of the form [14]

$$S^{(j)}_{ll'} = \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_+} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_-} \end{pmatrix} \tag{3.22}$$

where $\delta_\pm$ are the eigenphases, describing the scattering of a spin-0 by a spin-$\frac{1}{2}$ particle when the process is characterized by the total angular momentum $j = l \pm \frac{1}{2}$. Using the results of [10], employing the asymptotic properties of the solution of (3.21), the diagonal elements of the S-matrix turn out to be

$$S^{(\lambda)}(k) = \mathrm{e}^{2\mathrm{i}\delta_\lambda(k)} = -\frac{\lambda}{|\lambda|}\mathrm{e}^{\mathrm{i}\pi|\lambda|}\frac{\Gamma(\lambda + \frac{1}{2} - \mathrm{i}k)\Gamma(\frac{1}{2} + \mathrm{i}k)}{\Gamma(\lambda + \frac{1}{2} + \mathrm{i}k)\Gamma(\frac{1}{2} - \mathrm{i}k)} \tag{3.23}$$

where we have reverted to the usual notation using λ in the S-matrix as defined by (3.17). Note, that $\delta_\pm$ corresponds to the case with $\lambda = \pm(j + \frac{1}{2})$, respectively.

It is important to realize that the eigenphase shifts are not independent. This can be traced back to the fact that the transformation $\lambda \mapsto -\lambda$ amounts to calculating the phase shifts of potentials that are supersymmetry partners of each other [10]. For such partner potentials the reflection amplitude is the same. Since the S-matrix is related to the reflection amplitude by a multiplication factor $\mathrm{e}^{\mathrm{i}\pi l}$ which equals to $-\mathrm{e}^{\mathrm{i}\pi\lambda}$ if $\lambda > 0$ and to $\mathrm{e}^{\mathrm{i}\pi\lambda}$ if $\lambda < 0$, we can see that the S-matrix elements are the same up to a sign. To see this explicitly we have to use the reflection formula for the gamma function $\Gamma(z)\Gamma(1 - z) = \pi/\sin\pi z$ with $z = \lambda + \frac{1}{2} + \mathrm{i}k$. Since $\sin\pi z = \sin\pi(\lambda + \frac{1}{2})\cosh\pi k + \mathrm{i}\cos\pi(\lambda + \frac{1}{2})\sinh\pi k$ and $\lambda + \frac{1}{2}$ is half-integer, the imaginary part is zero. Hence, for this special choice $\Gamma(z)\Gamma(1 - z) = \Gamma(\bar{z})\Gamma(1 - \bar{z})$ yielding the identity

$$\frac{\Gamma(\lambda + \frac{1}{2} - \mathrm{i}k)}{\Gamma(\lambda + \frac{1}{2} + \mathrm{i}k)} = \frac{\Gamma(-\lambda + \frac{1}{2} - \mathrm{i}k)}{\Gamma(-\lambda + \frac{1}{2} + \mathrm{i}k)} \tag{3.24}$$

so the reflection amplitudes are the same for both of the cases $\lambda = \pm(j + \frac{1}{2})$.

Having shown that in the canonically transformed realization we can obtain solvable three-dimensional scattering problems with LS terms, in the next section we transform back to the non-local realization of (2.12) (involving the operator P^{-2}) in order to obtain non-local modified Coulomb potentials.

4. Non-local potentials with $\boldsymbol{LS}$ terms

The easiest way to obtain scattering potentials using the non-local realization of (2.12) is to use the inverse of the canonical transformation (2.6) *immediately in the (3.1b) expression for the Casimir operator* $\mathcal{C}'$. The inverse canonical transformation in this case yields

$$\mathcal{C}' = \frac{1}{\sqrt{2E}}\left(\frac{2\mathrm{i}}{P^2}\mathcal{M}\boldsymbol{SP} - \boldsymbol{SR}\right)\left(\frac{P^2}{2} - E\right). \tag{4.1}$$

Since the term $\frac{1}{2}P^2 - E$ is just *minus* the interaction term V, all we have to do is to evaluate (4.1) on the scattering states $|\pm\frac{1}{2}, \mathrm{i}f\rangle$ yielding by virtue of (3.7b) the constant $\pm\frac{1}{2}f$ for the left-hand side, and then inverting the operator standing before V. Note, that the states $|\frac{1}{2}, \mathrm{i}f\rangle$ and $|-\frac{1}{2}, \mathrm{i}f\rangle$ belong to representations mirror-conjugated to each other. Such representations are discriminated by $\mathcal{C}'$ which is a pseudoscalar operator; hence it is odd with respect to parity [15]. By realizing that

$$2\mathrm{i}\mathcal{M} = R[\sigma\boldsymbol{n}, \sigma\boldsymbol{P}] \tag{4.2}$$

which is an identity which can be proved by calculating the commutators $[\boldsymbol{SR}, \boldsymbol{SP}]$ and $[\boldsymbol{SP}, R]$ with $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$, and using the definition (3.2), one can convince oneself that

$$\mathcal{C}' = \frac{1}{2\sqrt{2E}} R(\boldsymbol{\sigma P})(\boldsymbol{\sigma n})(\boldsymbol{\sigma P}) \frac{1}{P^2}\left(E - \frac{P^2}{2}\right). \tag{4.3}$$

Acting with C' on the scattering states $|\pm\frac{1}{2}, \mathrm{i}f\rangle$ one can get the equation

$$\pm f|\pm\tfrac{1}{2}, \mathrm{i}f\rangle = \frac{1}{\sqrt{2E}} R(\boldsymbol{\sigma P})(\boldsymbol{\sigma n})(\boldsymbol{\sigma P})P^{-2}V|\pm\tfrac{1}{2}, \mathrm{i}f\rangle \tag{4.4}$$

where, according to section 1, the $so(3,1)$ labels f and k are related in a non-trivial way through the relation $f = \eta = Z_1Z_2e^2/k$ where $2E = k^2$. The inverse of the operator standing before V can readily be constructed thanks to the relations $(\boldsymbol{\sigma n})^2 = I$ and $(\boldsymbol{\sigma P})^2 = P^2$. The result for the interaction term V can be cast into the form

$$V = Z_1Z_2e^2(\boldsymbol{\sigma P})(\boldsymbol{\sigma n})(\boldsymbol{\sigma P})P^{-2}R^{-1}(\pm) \tag{4.5}$$

where the symbol $(\pm) \equiv \frac{1}{2}f\mathcal{C}'$ with the property $(\pm)^2 = I$ indicates whether we should evaluate V on a state or on its mirror-conjugated counterpart. Hence we can regard $(\pm)$ as a parity odd operator commuting with $\boldsymbol{J}$. Having this in mind, we can see from (4.5) that V is a *parity even* operator. In order to further specialize V we write it as $V = (P^2R)^{-1}(\sigma\boldsymbol{P})(\sigma\boldsymbol{n})(\sigma\boldsymbol{P})\alpha$, which can be verified by direct calculation. Here (the operator) $\alpha = (\pm)Z_1Z_2e^2$. Now we express $(\boldsymbol{\sigma n})(\boldsymbol{\sigma P})$ from (4.2) in terms of $\mathcal{M}$. After some algebra we obtain

$$V = R^{-1}(\boldsymbol{\sigma n})\alpha - 2R^{-1}P^{-2}R^{-1}\left(\frac{\partial}{\partial R} + \frac{\mathcal{M}}{\mathcal{R}}\right)\mathcal{M}(\boldsymbol{\sigma n})\alpha. \tag{4.6}$$

In order to further specialize V, as a next step we derive the radial Schrödinger equation. First, we define

$$\Psi_{\frac{1}{2},\mathrm{i}f,j\mp\frac{1}{2},j,m}(\boldsymbol{R}) \equiv \mathcal{R}_{\frac{1}{2},\mathrm{i}f,j\mp\frac{1}{2},j,}(R)\mathcal{Y}_{j\mp\frac{1}{2},j,m}(\boldsymbol{n}) \tag{4.7}$$

where $\boldsymbol{R} \equiv R\boldsymbol{n}$. The mirror-conjugated wavefunction transforming with respect to the representation $(-\frac{1}{2}, \mathrm{i}k)$ can be obtained from this by changing $(j \mp \frac{1}{2})$ to $(j \pm \frac{1}{2})$ on the left-hand side of equation (4.7).

Moreover, we recall that

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{R}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m'=-l}^{l} \frac{(R_<)^l}{(R_>)^{l+1}} Y^*_{lm'}(\boldsymbol{n}')Y_{lm'}(\boldsymbol{n}). \tag{4.8}$$

Using (4.8), (2.13) and the orthogonality of the functions $Y_{lm'}(\boldsymbol{n})$, we obtain

$$(P^{-2}\Psi_{\mp})(\boldsymbol{R}) = \frac{1}{(2j+1)\pm 1}\mathcal{Y}_{j\mp\frac{1}{2},j,m}(\boldsymbol{n}) \int \mathrm{d}R'(R')^2 \frac{(R_<)^{j\mp\frac{1}{2}}}{(R_>)^{(j\mp\frac{1}{2})+1}} \mathcal{R}_{\mp}(R') \tag{4.9}$$

where we have supressed the extra labels of Ψ and $\mathcal{R}$.

Now we would like to calculate the matrix elements of the (4.6) interaction term V in the basis spanned by the spinor harmonics $\mathcal{Y}_{j\mp\frac{1}{2},j,m}(\boldsymbol{n})$. Since V contains the term $\boldsymbol{\sigma n}$, we will need its action on the spinor harmonics $\mathcal{Y}_{j\mp\frac{1}{2},j,m}(\boldsymbol{n})$. As a first step we refer to the result [16]

$$\boldsymbol{\sigma n}\mathcal{Y}_{j\mp\frac{1}{2},j,m}(\boldsymbol{n}) = -\mathcal{Y}_{j\pm\frac{1}{2},j,m}(\boldsymbol{n}). \tag{4.10}$$

Using the fact that $(\pm) \equiv (2/f)\mathcal{C}'$ was introduced as a parity odd operator with exactly the same properties as $\boldsymbol{\sigma n}$ and repeating the steps in [16] for this operator, we obtain equation (4.10) for $(\pm)$ too. (The trick is to calculate $\boldsymbol{MJ}$ in a coordinate system with J_3

along the z-axis, then $\theta = 0$ and $\mathcal{Y}_{j\mp\frac{1}{2},j,m}$ does not depend on any of the angles θ and φ. The operator $M_3 J_3$ on this acts as the operator $(Z_1 Z_2 e^2/\sqrt{2E})S_3$ proving our claim.) By virtue of these results and (3.16) one can see that in this base only the diagonal elements are different from zero. This should not come as a surprise because V is even with respect to parity. According to [14] the interaction term and the S-matrix is diagonal in such cases, implying that under the scattering process a no-flip from the channel with $l = j \pm \frac{1}{2}$ to the channel with $l = j \mp \frac{1}{2}$ occurs. These diagonal elements of V can be calculated from the integral $\int \mathrm{d}\boldsymbol{n}\, \mathcal{Y}^*_{j\pm\frac{1}{2},j,m}(\boldsymbol{n})(V\Psi_\pm)(R\boldsymbol{n})$. After straightforward calculation we obtain the following matrix,

$$\begin{pmatrix} (\alpha/R) + (j+\frac{1}{2})W_1^+ - (j+\frac{1}{2})^2 W_2^+ & 0 \\ 0 & (\alpha/R) - (j+\frac{1}{2})W_1^- - (j+\frac{1}{2})^2 W_2^- \end{pmatrix} \tag{4.11}$$

acting on the vector $(\mathcal{R}_+, \mathcal{R}_-)^{\mathrm{T}}$. The kernel of the non-local operators W_1 and W_2 is

$$w_1^{(\pm)}(R, R') \equiv \frac{2\alpha}{2l+1} \frac{(R_<)^l}{(R_>)^{l+1}} \frac{R'}{R} \frac{\mathrm{d}}{\mathrm{d}R'} \tag{4.12a}$$

$$w_2^{(\pm)}(R, R') \equiv \frac{2\alpha}{2l+1} \frac{(R_<)^l}{(R_>)^{l+1}} \frac{1}{R} \tag{4.12b}$$

where the values of l are $j \pm \frac{1}{2}$, for $W_1^{(\pm)}$ and $W_2^{(\pm)}$, respectively.

Hence, we have shown that a three-dimensional coupled-channel scattering problem can indeed be obtained by using a matrix-valued realization for the $so(3,1)$ algebra. Moreover, from (4.11) one can see that we have managed to obtain non-local terms modifying the Coulomb potential. From the matrix form of the interaction term (it has only diagonal elements) it is clear that it describes a scattering problem where parity is conserved.

5. Calculation of the S-matrix

Having demonstrated that non-local interaction terms can appear in the Hamiltonian by using matrix-valued realizations for the $so(3,1)$ algebra, we now proceed to calculate the S-matrix of the corresponding scattering problem. Scattering states are characterized as eigenstates of the operators (3.1a, b) with $\boldsymbol{P}$ and $\boldsymbol{R}$ replaced accordingly by using the inverse of the canonical transformation (2.6). Since this transformation leaves invariant relation (3.6), these states are labelled as $|\pm\frac{1}{2}, \mathrm{i}f\rangle$. Moreover, such irreducible representation spaces, by assumption, are also eigenspaces of energy E. We have already seen in the previous section that in order to get modified Coulomb potentials the relation between the label f of the irreps and E has to be related as $f = \eta = Z_1 Z_2 e^2/k$ where $2E = k^2$.

The calculation will be performed in momentum space where

$$\Phi(\boldsymbol{P}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{e}^{\mathrm{i}\boldsymbol{P}\boldsymbol{R}} \Psi(\boldsymbol{R}). \tag{5.1}$$

The Casimir operator $\mathcal{C}$, not discriminating between the states $(\pm\frac{1}{2}, \mathrm{i}f)$, is

$$\mathcal{C} = -\left[\frac{\boldsymbol{R}}{\sqrt{2E}}\left(E - \frac{P^2}{2}\right)\right]^2 + \left(\mathcal{M} + \frac{P^2}{2E}\right)\frac{P^2 - 2E}{2P^2} - \frac{3}{4}. \tag{5.2}$$

After a scale transformation by $T = \mathrm{e}^{\frac{1}{2}\mathrm{i}\ln(2E)\boldsymbol{RP}}$, one can obtain from the eigenvalue problem of the Casimir operator the equation

$$\left(\frac{1-P^2}{4}\frac{\partial^2}{\partial P^2} + \frac{4P^4-5P^2+1}{2P}\frac{\partial}{\partial P}\right)\Phi + \left(\frac{P^2-1}{2P^2}\left(\frac{1-P^2}{2}\boldsymbol{L}^2 + \mathcal{M} + 6P^2\right) + 1 + f^2\right)\Phi = 0. \tag{5.3}$$

Introducing the new variable $y = (1-P^2)^{-2}$ equation (5.3) can be written in the form

$$\left(y(y-1)\frac{\partial^2}{\partial y^2} + \left(\frac{5}{2} - y\right)\frac{\partial}{\partial y} - \frac{3}{y} + \frac{\mathcal{M}(\mathcal{M}-1)}{4y} - \frac{\mathcal{M}(\mathcal{M}+1)}{4(y-1)} + 1 + f^2\right)\Phi = 0. \tag{5.4}$$

After separating the angular part of $\boldsymbol{P}$ in the usual way, we employ another similarity transformation

$$W = \left(\frac{1-y}{y}\right)^{-\frac{1}{2}l-2}(1-y)^{j+2} \tag{5.5}$$

transforming the radial part of (5.4) to the differential equation for the hypergeometric functions

$$\left(y(y-1)\frac{\partial^2}{\partial y^2} + (C - Dy)\frac{\partial}{\partial y} - AB\right)W^{-1}\mathcal{R}(P(y)) = 0 \tag{5.6}$$

where $D = A + B + 1$ and

$$A = B^* = j + 1 + \mathrm{i}|f| \qquad C = l + \tfrac{3}{2}. \tag{5.7}$$

Keeping track of all of our transformations made, we finally obtain the radial part of the original equation involving $\mathcal{C}$ of (5.2):

$$\mathcal{R}_{\frac{1}{2},\mathrm{i}f,l,j,}(P) = \left(\frac{P}{k}\right)^{-l-4}\left(-\frac{P^2/k^2}{P^2/k^2-1}\right)^{j+2} {}_2F_1\left(A, B, C, -\frac{1}{P^2/k^2-1}\right). \tag{5.8}$$

We have kept from the two linearly independent solutions of (5.7) in the neighbourhood of the singular point 0 the one for which $\mathcal{R}_{\frac{1}{2},\mathrm{i}f,l,j,}(0) = 0$ for $l \neq 0$. Note that the dependence of $\mathcal{R}$ on f manifests itself through the appearance of A and B of (5.7) in the argument of the hypergeometric function.

In order to obtain the S-matrix from the momentum space representation of the radial part of the wavefunction, we have to compare it with the Fourier transform of the wavefunction with modified Coulomb asymptotic behaviour. The latter is known to be [17]

$$\lim_{P\to k}\mathcal{R}_l^{\mathrm{Coulomb}}(P) \mapsto \left(1 - \frac{P^2}{k^2}\right)^{-1-\mathrm{i}f} + \mathrm{e}^{2\mathrm{i}\delta_l}2^{-4\mathrm{i}f}\frac{\Gamma(-\mathrm{i}f)}{\Gamma(\mathrm{i}f)}\left(1 - \frac{P^2}{k^2}\right)^{-1+\mathrm{i}f}. \tag{5.9}$$

This has to be compared with the radial part of the (5.8) momentum space solution in the limit $P \to k$, which for $|f| = f$ is given by (see [17] for the details of this limit)

$$\lim_{P\to k}\mathcal{R}_{l,j}(P) \mapsto \Gamma_1\left(1 - \frac{P^2}{k^2}\right)^{-1+\mathrm{i}f} + \Gamma_2\left(1 - \frac{P^2}{k^2}\right)^{-1-\mathrm{i}f} \tag{5.10}$$

where

$$\Gamma_{1,2} = \frac{\Gamma(l+\frac{3}{2})\Gamma(\mp 2\mathrm{i}f)}{\Gamma(j+1\mp\mathrm{i}f)\Gamma(l-j+\frac{1}{2}\mp\mathrm{i}f)}. \tag{5.11}$$

Here we have used the following property of hypergeometric functions [18],

$$F(A,B,C;z) = \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-z)^{-A}F(A,1-C+A,1-B+A;1/z) + \frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-z)^{-B}F(A,1-C+B,1-A+B;1/z) \tag{5.12}$$

valid for $|\arg(-z)| < \pi$. Comparing (5.9) and (5.10) gives

$$\mathcal{S}^{(\lambda)}(k) = e^{2i\delta_\lambda(k)} = -\frac{\lambda}{|\lambda|}e^{i\pi|\lambda|}\frac{\Gamma(\lambda+\frac{1}{2}+if)\Gamma(\frac{1}{2}-if)}{\Gamma(\lambda+\frac{1}{2}-if)\Gamma(\frac{1}{2}+if)} \tag{5.13}$$

where $f(k) = Z_1Z_2e^2/k$. The choice $|f| = -f$ gives the same result [17].

6. The geometrical meaning of matrix-valued realizations

Having shown that the non-standard realization (2.11) yields interesting exactly solvable scattering problems, in this section we will clarify its geometrical meaning. Moreover, we recall that in section 2 we postponed the discussion of the solution, $F(R) \equiv 1$, of equation (2.10). Here we would also like to discuss the connection of this solution to that with $F(R) \equiv 1/R^2$ yielding the solvable potentials of the previous sections.

As a first step, we notice that by employing a similarity transformation $\boldsymbol{O} \mapsto T^{-1}\boldsymbol{O}T$ to the operators $\boldsymbol{L}$ and $\boldsymbol{K}$ of (2.7) with $T(R) \equiv (1-R^2)^{-2}$, we obtain the operators

$$T^{-1}\boldsymbol{L}T = \boldsymbol{R}\times\boldsymbol{P} \qquad T^{-1}\boldsymbol{K}T = \tfrac{1}{2}(1+R^2)\boldsymbol{P} - \boldsymbol{R}(\boldsymbol{R}\boldsymbol{P}) \tag{6.1}$$

i.e. we have moved the differential operator $\boldsymbol{P}$ to the right. Next we introduce the six vectors $\boldsymbol{f}_j$ and $\boldsymbol{g}_j$, $j = 1, 2, 3$, with components

$$f_j^k(\boldsymbol{R}) = \varepsilon_{jlk}R_l \qquad g_j^k(\boldsymbol{R}) = \tfrac{1}{2}(1+R^2)\delta_{jk} - R_jR_k. \tag{6.2}$$

Hence the transformed operators of (6.1) can be written in the form of $f_j^kP_k$ and $g_j^kP_k$, respectively. Note, that the vectors $\boldsymbol{f}_j$ and $\boldsymbol{g}_j$ $j = 1, 2, 3$, generate the six infinitesimal transformations $\boldsymbol{R} \mapsto \boldsymbol{R}+\delta\boldsymbol{R}_j \equiv \boldsymbol{R}+\boldsymbol{f}(\boldsymbol{R})_j$ (similarly for $\boldsymbol{g}_j$), corresponding to the action of $so(3,1)$ on our coordinates.

Furthermore, we introduce the quantities

$$\boldsymbol{W}^f(\boldsymbol{R}) \equiv \boldsymbol{S} \qquad \boldsymbol{W}^g(\boldsymbol{R}) \equiv F(R)\boldsymbol{R}\times\boldsymbol{S} \tag{6.3}$$

where $F(R)$ is chosen as in (2.11) and $S_j \equiv \frac{1}{2}\sigma_j$. These are just the matrix-valued modifications of (2.11) (not affected by the similarity transformation $T(R)$). Hence, by virtue of (6.2) and (6.3), for the two possible sets of modified generators we have

$$T^{-1}J_jT = f_j^kP_k + W_j^f \qquad T^{-1}M_jT = g_j^kP_k + W_j^g. \tag{6.4}$$

In order to clarify the meaning of the matrix-valued modifications $\boldsymbol{W}^f$ and $\boldsymbol{W}^g$, we try to find an $su(2)$-valued vector field $\boldsymbol{A}_{\alpha\beta}(\boldsymbol{R}) \equiv \boldsymbol{A}^k(\boldsymbol{R})\sigma_{k\alpha\beta}$ satisfying the equations

$$f_j^k\partial_kA_i + (\partial_if_j^k)A_k = \partial_iW_j^f + i[A_i, W_j^f] \tag{6.5a}$$

$$g_j^k\partial_kA_i + (\partial_ig_j^k)A_k = \partial_iW_j^g + i[A_i, W_j^g] \tag{6.5b}$$

(matrix indices are left implicit). Notice, that the left-hand sides of equations (6.5) are the infinitesimal change in the vector field $\boldsymbol{A}$ under the coordinate transformations generated by the vectors $\boldsymbol{f}_j$ and $\boldsymbol{g}_j$. These quantities are just the Lie derivatives $\mathcal{L}_{\boldsymbol{f}_j}$ and $\mathcal{L}_{\boldsymbol{g}_j}$ of $\boldsymbol{A}$. The fact that the right-hand sides of (6.5) are *non-zero* means that our vector fields $\boldsymbol{A}$ are not invariant under the infinitesimal $so(3,1)$ transformations. Moreover, we recognize that the right-hand sides are just the *covariant derivatives* $D_j \equiv \partial_j + i[A_j, \,]$ of $\boldsymbol{W}^f$ and $\boldsymbol{W}^g$, with respect to $\boldsymbol{A}$. Hence $\boldsymbol{A}$ is an $su(2)$ *gauge potential*, and $D_i W_j^f$ and $D_i W_j^g$ are infinitesimal gauge transformations. Accordingly, equations (6.5) can be written in the compact form

$$\mathcal{L}_{f_j} A_i = D_i W_j^f \qquad \mathcal{L}_{g_j} A_i = D_i W_j^g \tag{6.6}$$

expressing the fact that the $\boldsymbol{A}$ we are looking for is a *symmetric (invariant) gauge field* [12, 18, 19], meaning that it is invariant under the infinitesimal $so(3,1)$ transformations only *up to an infinitesimal* $su(2)$ *gauge transformation*.

Now we try to find a solution to equations (6.5a, b). It is easy to see that the ansatz $\boldsymbol{A} \equiv G(R)\boldsymbol{S} \times \boldsymbol{R}$ solves (6.5a). (Note that, due to the fact that $\partial_k G(R) = G'(R)R_k/R$, no term containing $G'(R)$ occurs owing to the antisymmetry of ε_{jlk} in f_j^k. The sum of the remaining three terms proportional to $G(R)$ sums to zero according to the Jacobi identity for the matrices S_j.) This solution of (6.5a) means that we have found a class of rotationally ($SO(3)$) invariant $su(2)$-valued gauge fields parametrized by the arbitrary function $G(R)$. However, we are more ambitious in wanting to find a gauge field invariant under the larger group $SO(3,1)$. This can be done by further specifying the function $G(R)$ with the help of equation (6.5b). Using the ansatz $\boldsymbol{A} \equiv G(R)\boldsymbol{S} \times \boldsymbol{R}$ this equation yields terms proportional to $\varepsilon_{ijl}S_l$, $\varepsilon_{ijl}R_l(\boldsymbol{RS})$ and $R_i(\boldsymbol{S} \times \boldsymbol{R})_j$. The coefficients of these terms have to vanish independently, yielding the following set of three equations,

$$\tfrac{1}{2}(R^2-1)G = F + RF' \qquad (1-F)RG = F' \qquad \tfrac{1}{2}(1-R^2)G' - RG = F' \tag{6.7}$$

where according to (2.11) we have two possibilities for $F(R)$. From these equations we see that we obtain the solutions

$$\boldsymbol{A} = \frac{2}{R^2-1}\boldsymbol{S} \times \boldsymbol{R} \qquad \text{for } F(R) = 1 \tag{6.8a}$$

$$\boldsymbol{A}^{\mathcal{U}} = \frac{2}{R^2(1-R^2)}\boldsymbol{S} \times \boldsymbol{R} \qquad \text{for } F(R) = \frac{1}{R^2} \tag{6.8b}$$

where the notation $\boldsymbol{A}^{\mathcal{U}}$ suggests that we might be able to relate $\boldsymbol{A}^{\mathcal{U}}$ to $\boldsymbol{A}$ by an $SU(2)$ *gauge transformation* of the form

$$\boldsymbol{A}^{\mathcal{U}} = \mathcal{U}^\dagger \boldsymbol{A}\mathcal{U} - \mathrm{i}\mathcal{U}^\dagger \nabla \mathcal{U}. \tag{6.9}$$

To show that our expectation is really justified, note that

$$\boldsymbol{A} = \mathrm{i}\frac{R^2}{1-R^2}\boldsymbol{\sigma n}\nabla\boldsymbol{\sigma n} \tag{6.10}$$

where $\boldsymbol{n} \equiv \boldsymbol{R}/R$. Hence, with the choice $\mathcal{U}^\dagger \equiv \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi\boldsymbol{\sigma n}} = \mathrm{i}\boldsymbol{\sigma n}$ we can satisfy (6.9).

In order to gain further insight into the meaning of the non-Abelian gauge fields $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathcal{U}}$, we also introduce the field strength F_{jk}

$$F_{jk} \equiv \partial_j A_k - \partial_k A_j + \mathrm{i}[A_j, A_k] \tag{6.11}$$

transforming covariantly (i.e. $F_{jk}^{\mathcal{U}} = \mathcal{U}^\dagger F_{jk}\mathcal{U}$) under an $SU(2)$ gauge transformation. Note also that $F_{jk} \to 0$ for $R \to \infty$, since $A_j \to \mathrm{i}\mathcal{U}^\dagger\partial_j\mathcal{U}$ for $R \to \infty$. One can easily show [19, 20] that with the help of F_{jk} equation (6.6) can be written in the form

$$f_j^k F_{ki} = D_i \Phi_j^f \qquad g_j^k F_{ki} = D_i \Phi_j^g \tag{6.12}$$

where

$$\Phi_j^f = W_j^f - f_j{}^k A_k \qquad \Phi_j^g = W_j^g - g_j^k A_k. \tag{6.13}$$

It is important to realize that the quantities Φ_j^f and Φ_j^g transform covariantly under an $SU(2)$ gauge transformation, as can be seen from equation (6.12) by virtue of the covariant transformation properties of F_{jk} and D_j. With the help of Φ_j^f and Φ_j^g the modified generators of (6.4) can be written in the instructive form

$$T^{-1}J_jT = f_j^k(P_k + A_k) + \Phi_j^f \qquad T^{-1}M_jT = g_j^k(P_k + A_k) + \Phi_j^g. \tag{6.14}$$

Since $\boldsymbol{P} + \boldsymbol{A} \equiv -\mathrm{i}\boldsymbol{D}$, we can see that our modified generators transform covariantly too.

Now we are in the position to clarify the relationship between the two solutions in equation (2.11). For the two possible choices $F(R) = 1$ and $F(R) = 1/R^2$ we have two possible sets of modified $so(3,1)$ generators. According to (6.8a, b) and (6.14) we can see that

$$(T^{-1}J_jT)^{\mathcal{U}} = \mathcal{U}^\dagger(T^{-1}J_jT)\mathcal{U} \qquad (T^{-1}M_jT)^{\mathcal{U}} = \mathcal{U}^\dagger(T^{-1}M_jT)\mathcal{U} \tag{6.15}$$

i.e. the two possible solutions are related to each other by the $su(2)$ gauge transformation $\mathcal{U} \equiv \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\boldsymbol{\sigma n}} = -\mathrm{i}\boldsymbol{\sigma n}$.

An immediate consequence of this important result is that we can easily derive the scattering potential corresponding to the solution $F(R) = 1$ by simply gauge transforming the (3.14) Schrödinger equation of section 3 obtained for the realization based on $F(R) = 1/R^2$. Indeed, the Casimir operators (3.1a, b) are also gauge covariant; hence the new scattering Hamiltonian can be obtained by gauge transforming with $\mathcal{U}$. (Note that $\mathcal{U}(\boldsymbol{n})$, hence the derivatives with respect to $r(R)$ of (3.10), in the kinetic term of (3.14) are not transforming.) By virtue of (3.4), $\boldsymbol{\sigma n}\mathcal{M}\boldsymbol{\sigma n} = -\mathcal{M}$ and $\boldsymbol{\sigma n}\boldsymbol{L}^2\boldsymbol{\sigma n} = \boldsymbol{L}^2 + 2\mathcal{M}$; hence the gauge transformed version of the (3.14) Schrödinger equation is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\boldsymbol{L}^2}{\sinh^2 r} + \frac{\mathcal{M}}{2\sinh^2(r/2)}\right)\Psi_{\frac{1}{2},\mathrm{i}k,\lambda} = k^2\Psi_{\frac{1}{2},\mathrm{i}k,\lambda} \tag{6.16}$$

yielding the scattering potential

$$V^{\mathcal{U}} = -\frac{l(l+1)}{2}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right) + \frac{\lambda}{4\sinh^2(r/2)}. \tag{6.17}$$

This potential yields the same (3.21) form for the radial equation with the choice $\lambda \mapsto -\lambda$. Hence the roles of the eigenphases have to be changed in the S-matrix. This can also be seen by noting that by virtue of equation (4.10) $\mathcal{U} = -\mathrm{i}\boldsymbol{\sigma n}$ is just the matrix $\mathrm{i}\sigma_1$ on the (3.15) states. It follows that $S^{j^{\mathcal{U}}} = \sigma_1 S^j \sigma_1$; hence

$$S^{(\lambda)^{\mathcal{U}}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_+} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_-} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_-} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_+} \end{pmatrix} \tag{6.18}$$

where the eigenphases $\delta_\pm$ are defined by equation (3.23).

Of course the example above served merely illustrative purposes. By using $su(2)$ gauge transformations of a more general kind we can derive a large number of interaction terms. The author is intending to elaborate this promising idea in a subsequent publication.

7. Conclusions

In this paper we have shown that by considering matrix-valued differential operators for the realization of the $so(3,1)$ algebra one can obtain non-local potentials with LS terms. Such realizations modify quite naturally the usual realization of the $so(3,1)$ algebra in terms of the angular momentum and the Runge–Lenz vector. We have obtained potentials of

Pöschl–Teller type and, after employing a canonical transformation, non-local potentials. The corresponding *S*-matrices are also calculated; for the non-local potential we used the momentum representation. We have also stressed the important role played by Casimir operators *other* than the quadratic Casimir for the identification of the scattering states. The fact that these operators are non-zero (unlike for the usual cases) enabled us to calculate the interaction terms more quickly.

The geometrical meaning of our realization has also been clarified by showing that the generators can be rewritten in covariant form. In this case the derivatives are replaced by covariant derivatives by using $su(2)$-valued gauge fields. Hence, by employing $su(2)$-valued gauge transformations one can obtain a whole class of $so(3,1)$ realizations. This observation has important implications. One can derive a large number of solvable potentials by simply gauge transforming the Hamiltonian of the scattering problem. A simple example of this procedure was given in section 6. One can show [10] that the potentials related to each other by the $su(2)$ gauge transformation of the form $\mathcal{U} \equiv \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\boldsymbol{\sigma n}} = -\mathrm{i}\boldsymbol{\sigma n}$ (see potentials (3.20) and (6.17)) are supersymmetry partners of each other. Hence, we have also found an interesting relationship between gauge transformations and SUSY transformations.

Finally, we comment on a possible generalization of this construction for the algebra $so(3,2)$ which is frequently used in realistic models of heavy-ion reactions [9, 21, 22]. Our construction of matrix-valued realizations was based on an $su(2) \sim so(3)$ irreducible representation. In this paper we used merely the simplest non-trivial, spin-$\frac{1}{2}$, representation. One can show that realization (2.11) is just the induced representation for $so(3,1)$ induced by the above-mentioned $so(3)$ representation. This representation is expressed with the help of the three coordinates (R_1, R_2, R_3) which are stereographically projected coordinates of the hyperboloid $-X_1^2 - X_2^2 - X_3^2 + X_4^2 = 1$ which is the coset $SO(3,1)/SO(3)$. This coordinate transformation is of the form

$$\boldsymbol{X} = \frac{2}{1-R^2}\boldsymbol{R} \qquad X_4 = \frac{1+R^2}{1-R^2}. \tag{7.1}$$

In the same spirit one can try to use the (2.11) $so(3,1)$ realization to obtain an $so(3,2)$ realization. In this case we have to express the ten $so(3,2)$ generators generating the isometries of the hypersurface $-X_1^2 - X_2^2 - X_3^2 + X_4^2 + X_5^2 = 1$ in terms of the coordinates (R_1, R_2, R_3, χ):

$$\boldsymbol{X} = \frac{2}{1-R^2}\boldsymbol{R} \qquad X_4 = \frac{1+R^2}{1-R^2}\cos\chi \qquad X_5 = \frac{1+R^2}{1-R^2}\sin\chi. \tag{7.2}$$

Then we can express the $so(3,2)$ generators in terms of the usual generators $\boldsymbol{L}$ and $\boldsymbol{K}$, and then try to use the modified generators $\boldsymbol{J}$ and $\boldsymbol{M}$ instead. The construction is based, in this case, on the coset $SO(3,2)/SO(3,1)$ which can be parametrized by the aforementioned four coordinates. Notice that in this case the inducing finite-dimensional matrix representation of $so(3,1)$ must be non-unitary. In this case the matrix part of the generators is non-Hermitian. This could be the way to obtain non-Hermitian interaction terms, hence arriving at a group theoretical description of optical potentials. Such ideas will be investigated in a forthcoming publication.

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